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PRELIMINARY ORBIT DETERMINATION OF ARTIFICIAL EARTH SATELLITES FROM A SMALL NUMBER OF ANGLE-ONLY OBSERVATIONS

Barry W. Bryant, et al

Massachusetts Institute of Technology

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13. ABSTRACT

The problem of determining the preliminary orbital elements of a previously unobserved earth satellite from a minimum number of angle-only observations has been investigated.

The Planetary Ephemeris Program (PEP) was used to generate exact ephemerides (of selected orbits) which were then used as input to an angle-only tracking algorithm derived from the method of Gauss. This algorithm yields the approximate orbital elements under the assumption of Keplerian motion. Comparison of these elements with the ones assumed in PEP has indicated that the Keplerian approximation improves considerably as the unknown satellite altitude is increased from 1/4 to 3 times synchronous, and that such a tracking algorithm would be useful in many practical situations where one is constrained to a few angle-only observations.

Estimates are given of the accuracy needed in the input data to insure reasonable success in this method of preliminary orbit determination.

14. KEY WORDS

orbit determination | earth satellite | PEP (planetary ephemeris program)

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PREFACE

This report represents work carried out in the Spring of 1972 to determine whether or not it is possible to track one earth satellite from another, given the spacial coordinates of the <u>observing</u> satellite and the angular coordinates of the <u>observed</u> at specified times . The present study is not intended to be exhaustive, but rather is intended to establish the feasibility of preliminary earth satellite orbit determination from a small number of angle-only observations.

The difficult and important questions of detection and discrimination against the celestial background will not be discussed. It will be assumed that these difficulties can be surmounted and that the necessary observations can be made.

To make the problem tractable, we have chosen a few simple examples of orbit configurations which were studied in some detail. The Planetary Ephemeris Program[1] (PEP) has been particularly useful in generating the ephemerides needed in the method of preliminary orbit determination. This extensive computer program integrates the motion, and the partial derivatives of the motion over many orbital periods about a non-spherical Earth; also, it takes into account various other perturbations and with subsequent least squares analyses leads

to a maximum likelihood orbit determination. The object of this study is to determine the osculating orbital elements from limited amounts (less than one period) of angle-only data and to compare these with the values assumed in the general PEP calculations.

Historically, the problem of finding the orbit of a satellite from a limited number of angle-only observations is very interesting. The following brief account will serve as an illustration. On January 1, 1801, Piazzi discovered the asteroid, Ceres, after which he soon became ill and had to cease his observations. The asteroid thereafter moved near the sun where all observations became impossible. It was feared that the minor planet would be lost. The methods of Laplace and Lagrange lacked the required precision to locate such an object months after its first observation. It remained for Gauss to work out a method, later completely revised and generalized[2], which allowed the rediscovery of Ceres on the night of December 7, 1801 by DeZach. It is Gauss' revised method which we have used to determine the approximate orbital elements. A detailed discussion of it is given for reference and for completeness. A computer program based on this method was written by one of the authors (DME) in order to carry out the necessary computations.

It has been assumed throughout that it is not practical to follow a given satellite over more than a fraction of its period. This is a basic constraint of the problem; otherwise PEP or some similar procedure could be applied directly without resort to the intermediate step of preliminary orbit determination. It is to be understood that once a preliminary orbit has been determined, refined methods can then be employed on subsequent observations (which are not necessarily contiguous in time but which can be unambiguously associated with that orbit) to arrive at a precise orbit.

I. INTRODUCTION

The problem of accurately predicting the position of a body in its orbit based on a minimum number of observations is, in general, insoluble analytically. In many cases a two-body approximation is unrealistic. For example, to analyze the motions within a star cluster, the theory of General Perturbations (numerical integration) would have to be applied. On the other hand, most Solar System problems like the Earth-satellite problem, can be handled by the techniques of Special Perturbations. Here one assumes that the force function is of the form U + R where U represents the two-body approximation and R the disturbing function. In general, the contributions from R must be small compared to U. Unfortunately, to employ either technique one needs either many observations covering at least one period or, equivalently, the elements of the osculating orbit. Assuming that this is not the case, one is forced to neglect all perturbations and assume simple Keplerian motion. Hopefully the two-body approximation will at least yield enough precision to enable future observations to be made.

Six elements uniquely define an orbit and the position of the body describing the orbit. Three of these define the orientation in space, two of them define the size and shape of the orbit and the sixth will give the position of the body within

the orbit. In the classical case of a planet in an elliptical orbit about the Sun, the elements are defined with respect to the ecliptic and the First Point of Aries. See Figure 1. Briefly, the six elements are: the semi-major axis, a; the eccentricity, e; the inclination of the plane of the orbit to the plane of the ecliptic, i; the angle that the major axis makes with the line of nodes (the longitude of perihelion), ω; the angle that the line of nodes makes with the line from the Sun to the vernal equinox, (the longitude of the ascending node), Ω; and the time of perihelion passage, τ. Six elements have to be found; hence, three observations of the body's right ascension, and declination at three different times constitute the minimum data for orbit determination.

From this preliminary orbit it is possible to compile a table of predicted positions, an ephemeris, to be used for tracking the body so that future observations may be made. Additional observations would then be used presumedly, to improve the orbit. There are many ways to accomplish this and only the basic ideas will be presented here.

Assume the elements of the preliminary orbit, σ_i, i = 1,...6 have been calculated for a geocentric satellite. Because all possible perturbations were neglected, these elements are not the elements of the actual orbit. Hence, any observed quantity ϕ where

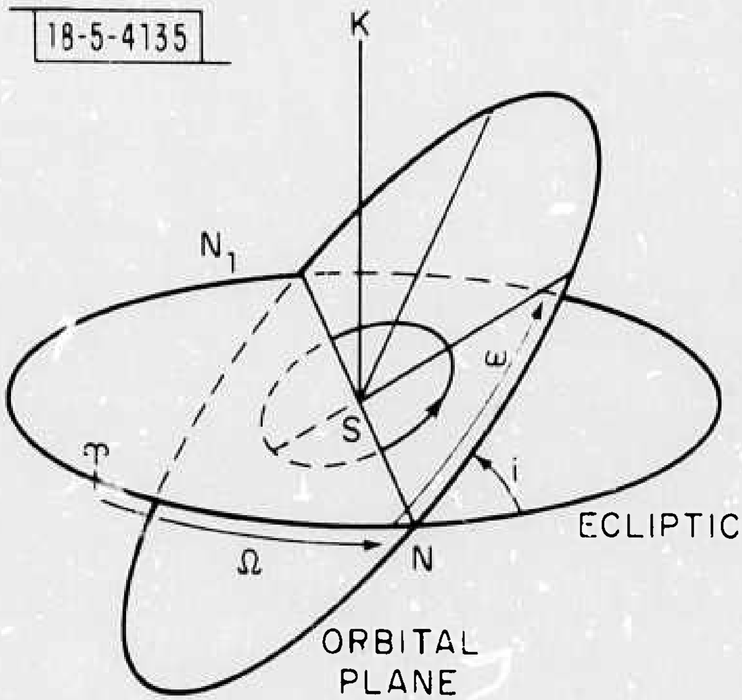


Fig. 1. The orientation of an orbit in space. In the earth satellite case, the plane of the earth's equator replaces the ecliptic.

$$\phi = \phi(\sigma_i, q_i, t) \quad i = 1, \ldots 6$$

and the q_i's are the six elements of the Earth's orbit, will differ from the predicted quantities ϕ_p at a given time. The ϕ's may be observations of the right ascension and declination (α, δ), the geocentric co-ordinates (x,y,z) or the velocity components $(\dot{x},\dot{y},\dot{z})$, etc. Assuming the preliminary orbit was reasonably close to the actual one, the change in the orbital elements $\Delta\sigma_i$ will be slight. The change in ϕ will be

$$\Delta\phi = \sum_{i=1}^{6} \frac{\partial\phi}{\partial\sigma_i} \Delta\sigma_i .$$

Then if we let the difference between the observed quantity and the predicted quantity be

$$\Delta\phi = \phi_o - \phi_p$$

we have,

$$\Delta\phi_j = (\phi_o - \phi_p)_j = \left(\sum_{i=1}^{6} \frac{\partial\phi}{\partial\sigma_i}\right)_j \Delta\sigma_i \quad j = 1, \ldots n$$

where the subscript j means the quantity was observed or predicted at time t_j.

If n = 6, the problem is a straightforward one of six equations in six unknowns. If $n > 6$, then the equations can be

solved by least squares or a similar technique. In either case the $\Delta\sigma_i$ can then be added to the σ_i to yield "improved" values of the orbital elements. Theoretically, the process can be carried out for all sets of future observations. If and only if the osculating elements are reasonably accurate can we hope to continually improve the orbital predictions.

II. THE METHOD OF GAUSS

Several methods have been developed to solve the problem of preliminary orbit determination. Needless to say, until the advent of high speed computers the orbits so determined were hardly to be considered preliminary. The basic methods of Laplace, of Lagrange and of Gauss are probably the most useful. The first two methods have been neglected in favor of Gauss' technique since the latter makes no assumptions about the time between observations and readily lends itself to an iterative procedure.

Let (x,y,z) denote a geocentric rectangular equatorial coordinate system as in Figure 2: the x-axis is directed towards the First Point of Aries. Let (ξ,η,ζ) be a parallel coordinate system centered on the Hunter satellite, H, then the unit vector $\hat{u}$ defining the Hunted satellite in the Hunter satellite reference system is

$$\hat{u} = (\cos\alpha\cos\delta)\,\hat{i} + (\sin\alpha\cos\delta)\,\hat{j} + \sin\delta\hat{k}$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along the ξ,η,ζ axes respectively and (α_i,δ_i), i = 1, 3 are the three observations of the observed satellite, h, from the observing satellite, H. From Figure 2 we see that

$$\hat{r}_i = \rho_i\hat{u}_i - \hat{R} \qquad (1)$$

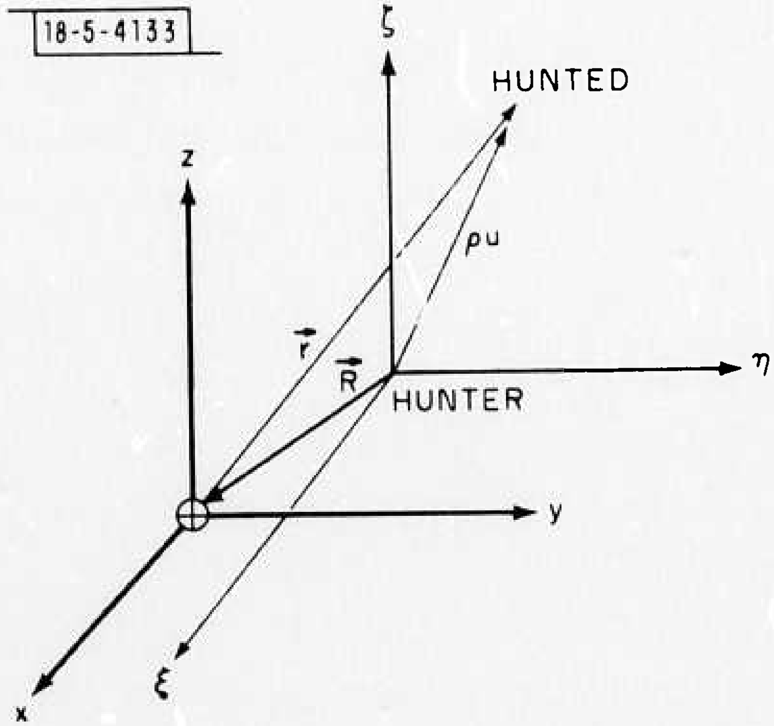


Fig. 2. The vector relationships resulting from a hunter satellite centered coordinate syste' ı.

where $\hat{R} = (X,Y,Z)$ is the position of the Earth as seen from the Hunter satellite and ρ is the distance to the observed satellite. We must assume that the equatorial coordinates of the Earth are known accurately at each of the three observation times.

Since we are assuming Keplerian motion we know that the motion takes place in a plane; hence, one of the unknown radius vectors may be written as a linear combination of the other two. We have

$$\hat{r}_2 = c_1\hat{r}_1 + c_3\hat{r}_3 \tag{2}$$

where the coefficients c_1, c_3 are the so called triangle ratios. If c_1 and c_3 are known, then substitution of equation (1) into equation (2) furnishes, in component form, three linearly independent equations in the unknowns ρ_1, ρ_2, and ρ_3, the hunter-hunted satellite distances at the times of the observations. Since the coefficients are unknown, we approximate then iterate. First we note that the magnitude of the vector cross product of any two of these radius vectors is double the area of the triangle formed by the Earth and the two positions of the Hunted satellite. See Figure 3. Hence, we can write the coefficients as ratios of triangle areas. Taking $\hat{r}_1$ crossed with equation (2), we have

$$\hat{r}_1 \times \hat{r}_2 = c_3\hat{r}_1 \times \hat{r}_3$$
$$c_3 = [r_1, r_2]/[r_1, r_3] \tag{3}$$

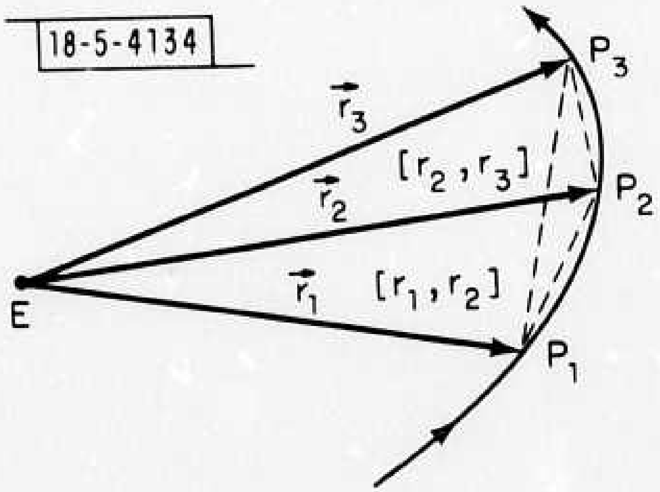


Fig. 3. The geometry of triangle - sector areas.

where the brackets denote triangle areas. Similarly by calculating $\hat{r}_2 x \hat{r}_3$ we find for the coefficient c_1:

$$c_1 = [r_2,r_3]/[r_1,r_3]. \tag{4}$$

Let (r_i,r_j) denote the area of the sector of the ellipse bounded by the radius vectors $\hat{r}_i,\hat{r}_j$ and denote the sector-triangle ratios by

$$\bar{y}_1 = \frac{(r_2,r_3)}{[r_2,r_3]} \quad \bar{y}_2 = \frac{(r_1,r_3)}{[r_1,r_3]} \quad \bar{y}_3 = \frac{(r_1,r_2)}{[r_1,r_2]}. \tag{5}$$

Then following Gauss we can rewrite equations (3) and (4) as sector-triangle ratios:

$$c_1 = \frac{(r_2,r_3)\bar{y}_2}{(r_1,r_3)\bar{y}_1} \qquad c_3 = \frac{(r_1,r_2)\bar{y}_2}{(r_1,r_3)\bar{y}_3}$$

or, recalling Kepler's second law, namely that the area described by the radius vector is proportional to the time,

$$c_1 = \frac{(t_3 - t_2)\bar{y}_2}{(t_3 - t_1)\bar{y}_1} \quad c_3 = \frac{(t_2 - t_1)\bar{y}_2}{(t_3 - t_1)\bar{y}_3}. \tag{6}$$

Encke[3] developed a series solution for the sector-triangle ratios; his expansion to the third order is,

$$\bar{y} = 1 + \frac{4m}{3} - \frac{88m^2}{45} - \frac{8m\ell}{5} + \frac{5312m^3}{945} + \frac{512m^2\ell}{105} + \frac{64m\ell^2}{35} + \ldots$$

where,

$$m = \eta \sec^3 \gamma \qquad \ell = \frac{(1 - \cos\gamma)}{2 \cos \gamma}$$

$$\eta = \mu k^2 \frac{(t_j - t_i)^2}{(r_i + r_j)^3} \qquad \cos(f_j - f_i) = \frac{(\hat{r}_i \cdot \hat{r}_j)}{(r_i r_j)}$$

$$\sec\gamma = \frac{r_i + r_j}{2(r_i r_j)^{1/2} \cos((f_j - f_i)/2))}$$

k is Gauss[†] constant, and f is the true anomaly (the angle swept out by r since the time of perigee). If we now let

$$\tau_1 = k\ (t_3 - t_2)$$

$$\tau_2 = k\ (t_3 - t_1)$$

$$\tau_3 = k\ (t_2 - t_1)$$

denote the time intervals between observations (in some appropriate units determined by k), we can express the coefficients in (6) as,

$$c_1 = \frac{\tau_1}{\tau_2} + \frac{\tau_1}{6\tau_2}\left[1 - \frac{\tau_1^2}{\tau_2^2}\right] \frac{\tau_2^2}{r_2^3} = a_1 + \frac{b_1}{r_2^3}$$

$$c_3 = \frac{\tau_3}{\tau_2} + \frac{\tau_3}{6\tau_2}\left[1 - \frac{\tau_3^2}{\tau_2^2}\right] \frac{\tau_2^2}{r_2^3} = a_3 + \frac{b_3}{r_2^3} \qquad (7)$$

to the first order in m. The coefficients a_1, b_1, a_3, and b_3 can be calculated immediately from the input data. Substituting

† See Appendix A

equation (1) into equation (2) we have

$$\rho_2\hat{u}_2 = c_1(\rho_2\hat{u}_1 - \hat{R}_1) + \hat{R}_2 + c_3(\rho_3\hat{u}_3 - \hat{R}_3) \tag{8}$$

then, substituting the expressions in (7) for the coefficients c_1, c_3,

$$\rho_2\hat{u}_2 = \left(a_1 + \frac{b_1}{r_2^3}\right)(\rho_2\hat{u}_1 - \hat{R}_1) + \hat{R}_2 + \left(a_3 + \frac{b_3}{r_2^3}\right)(\rho_3\hat{u}_3 - \hat{R}_3).$$

Operating on this equation with $(\cdot\hat{u}_1\times\hat{u}_3)$ yields, after a bit of arithmetic, an equation of the form:

$$\rho_2 = A + \frac{B}{r_2^3} \tag{9}$$

where,

$$A = \frac{a_1[\hat{R}_1\cdot\hat{u}_1\times\hat{u}_3] - [\hat{R}_2\cdot\hat{u}_1\times\hat{u}_3] + a_3[\hat{R}_3\cdot\hat{u}_1\times\hat{u}_3]}{[\hat{u}_1\cdot\hat{u}_2\times\hat{u}_3]}$$

$$B = \frac{b_1[\hat{R}_1\cdot\hat{u}_1\times\hat{u}_3] + b_3[\hat{R}_3\cdot\hat{u}_1\times\hat{u}_3]}{[\hat{u}_1\cdot\hat{u}_2\ \hat{u}_3]}. \tag{10}$$

A second equation relating ρ_2 and r_2 is evident from Figure 2. By the cosine law

$$r_2^2 = \rho_2^2 + R_2^2 - 2(\hat{u}_2\cdot\hat{R}_2). \tag{11}$$

We now have two equations, (9) and (11), in two unknowns which can, theoretically, be solved for ρ_2 and r_2. Any iterative procedure can be used in the solution provided a reasonable initial guess at r_2 can be made. Alternatively, one could form the following eighth order equation and then use Bairstow's Method[4] or an equivalent technique to find the roots. Substituting equation (9) for ρ_2 into equation (11) yields,

$$r_2^8 - (C + AD + A^2)\, r_2^6 - (BD + 2AB)\, r_2^3 - B^2 = 0$$

where $C = R_2^2$ and $D = -2(\hat{u}_2 \cdot \hat{R}_2)$. If the coefficients c_1 and c_3 are exact, then this equation will have a trivial root $r_2 = R_2$ at $\rho_2 = 0$. In any event, one of the roots will approximate the trivial solution and either four or six of the roots will be complex. The latter case yields no solution, since the remaining real root will be negative. In the first case, then, five of the roots can be immediately disregarded. It can also be shown by considering the number of variations in sign of the coefficients that of the remaining three real roots we must have either two positive and one negative or all three negative. Again, the case of three negative roots will yield no solution. Hence, there are two 'possible' roots to this equation to be considered. In a well defined case (i.e., when Keplerian motion is a good first approximation) one of these possible roots will

be outside of any reasonable limits to the problem. In the Earth satellite case, for example, obvious limits would be the radius of the Earth and 1 A.U. (astronomical unit). Hence, in a well defined case there is but one solution and one orbit for the satellite being observed.

When a solution has been found, the coefficients c_1 and c_3 can be evaluated from equations (7). Then operating on equation (8) first with $(\cdot\hat{u}_2\ \hat{u}_3)$ and then with $(\cdot\hat{u}_1\ \hat{u}_2)$ we find for the other two Hunter-Hunted distances:

$$\rho_1 = \frac{c_1[\hat{R}_1\cdot\hat{u}_2\times\hat{u}_3] - [\hat{R}_2\cdot\hat{u}_2\times\hat{u}_3] + c_3[\hat{R}_3\cdot\hat{u}_2\times\hat{u}_3]}{c_1[\hat{u}_1\cdot\hat{u}_2\times\hat{u}_3]}$$

$$\rho_3 = \frac{c_1[\hat{R}_1\cdot\hat{u}_1\times\hat{u}_2] - [\hat{R}_2\cdot\hat{u}_1\times\hat{u}_2] + c_3[\hat{R}_3\cdot\hat{u}_1\times\hat{u}_2]}{c_3[\hat{u}_3\cdot\hat{u}_1\times\hat{u}_2]}$$

Equation (1) will then give the two radius vectors $\hat{r}_1$ and $\hat{r}_3$.

Because the coefficients c_1 and c_3 were determined from truncated series expressions, the solution ρ_2, r_2 is only approximate. Hence, the solution might be improved by iterating on the coefficients. First the observation times and the time intervals should be corrected for light time by using the relation

$$t_i' = t_i - .577 \times 10^{-2} \rho_i \text{(days)}$$

where ρ_i is the distance (A.U.) calculated in the previous iteration. The sector-triangle ratios are calculated again using Encke's series solution including second and perhaps, third order terms and using the previously calculated radius vectors $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$. The "improved" coefficients will be:

$$c_1' = \frac{\tau_1' \bar{y}_2'}{\tau_2' \bar{y}_1'}$$

$$c_3' = \frac{\tau_3' \bar{y}_2'}{\tau_2' \bar{y}_3'}$$

If these coefficients agree to within the desired accuracy with the previously calculated values, then the iteration may be stopped. If not, then the coefficients in equation (7) should be recalculated according to:

$$a_1' = \tau_1'/\tau_2' \qquad a_3' = \tau_3'/\tau_2'$$

$$b_1' = a_1' \left(\frac{\bar{y}_2'}{\bar{y}_1'} - 1 \right) r_2^3 \qquad b_3' = a_3' \left(\frac{\bar{y}_2'}{\bar{y}_3'} - 1 \right) r_2^3$$

Equation (9) is reformed and a second solution with equation (11) is carried out. If the three observations are spread over a relatively small section of arc, then three iterations will probably prove satisfactory. With the final values of ρ_1 and ρ_3 the radius vectors $\hat{r}_1$ and $\hat{r}_3$ are calculated by equation (1) and with these the orbital elements are calculated.

III. CALCULATION OF THE ELEMENTS

Only two radius vectors and the times corresponding to these positions are needed to calculate the six orbital elements. In the heliocentric problem it is generally the ecliptic elements that are required. In that case, the radius vectors $\hat{r}_1 = (x_i, y_i, z_i)$ can be transformed by:

$$x' = x$$

$$y' = y\cos\varepsilon + z\sin\varepsilon$$

$$z' = -y\sin\varepsilon + z\cos\varepsilon$$

where ε is the mean obliquity of the ecliptic. In the geocentric problem the equator is considered the fundamental reference plane; hence no transformation is required.

Let $\hat{r}_1 = (x_1, y_1, z_1)$ and $\hat{r}_3 = (x_3, y_3, z_3)$; then the equation of the plane of the orbit will be

$$ax + by + z = 0$$

where,

$$a = \frac{y_1 z_3 - y_3 z_1}{x_1 y_3 - x_3 y_1} \qquad b = \frac{z_1 x_3 - z_3 x_1}{x_1 y_3 - x_3 y_1}$$

and the equation of the line of nodes, the intersection of the orbital plane and the plane of the equator will be

$$ax + by = 0.$$

But the slope of the line of nodes is just

$$\tan\Omega = -\frac{a}{b} = \frac{y_1 z_3 - y_3 z_1}{x_1 z_3 - x_3 z_1}$$

where Ω is the longitude of the ascending node. See Figure 1. The inclination of the orbital plane to the plane of the equator can be gotten from

$$|\cos i| = \frac{1}{[1 + a^2 + b^2]^{1/2}}$$

To determine the quadrants of Ω and i, first determine if the points (x_1, y_1) and (x_3, y_3) are in different quadrants. If so, then the direction of motion (direct or retrograde) is apparent. If the points are in the same quadrant, then when $y_3/x_3 > y_1/x_1$ the motion is direct ($i < 90°$); conversely the motion is indirect ($i > 90°$). The slope of the line of nodes will indicate through which quadrants the line passes; a positive slope indicates quadrants 1 and 3, while a negative slope indicates quadrants 2 and 4. To determine which of the two quadrants contains the ascending node, it is necessary to consider the quadrant and slope of the third radius vector with respect to the line of nodes. If the motion is direct and $y_3/x_3 > \tan\Omega$,

the satellite will pass through the ascending node next. If the motion is indirect and y_3/x_3 is greater than $\tan\Omega$, then the satellite will pass through the descending node before the ascending node. Thus, Ω and i are determined. These two elements orient the orbit in space.

Using Kepler's law of areas, we have that the sector area is,

$$2(r_1,r_3) = kp^{1/2}(t_3 - t_1)$$

where p is the semi-lattice rectum, $p = a(1 - e^2)$ and $kp^{1/2}/2$ is the magnitude of the area velocity. See Figure 4. For the triangle area we have

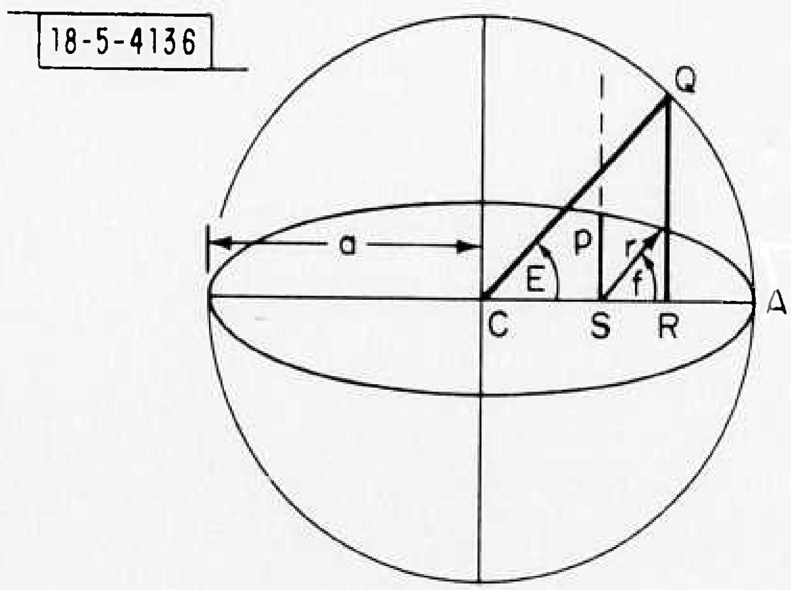
$$2[r_1,r_3] = r_1r_3\sin(f_3 - f_1)$$

where f is the true anomaly. Then using the sector triangle ratio $\bar{y}_2$ previously calculated, we can solve for p:

$$\bar{y}_2 = \frac{(r_1,r_3)}{[r_1,r_3]}$$

$$p^{1/2} = \frac{r_1r_3\sin(f_3 - f_1)\bar{y}_2}{k(t_3 - t_1)} .$$

To find the eccentricity, e, we use the equation of a conic section, namely,



$$SR = r \cos f$$
$$r \cos f = a(\cos E - e)$$
$$e = CS/CA$$

Fig. 4. The relationship between radius vector, $\hat{r}$ the semi-lattice rectum p, the true anomaly f, the eccentric anomaly E, the semi-major axis a, and the eccentricity e.

$$r = \frac{p}{1 + e\cos f}.$$

Setting $q_1 = p/r_1 - 1$, we form the first of two equations in two unknowns, e and f_1:

$$e\cos f_1 = q_1. \tag{12}$$

Similarly,

$$e\cos f_3 = q_3 = p/r_3 - 1.$$

Since we can calculate the difference in anomalies from $\cos(f_3 - f_1) = (\hat{r}_1 \cdot \hat{r}_3)/(r_1 r_3)$, we could rewrite f_3 as $f_1 + (f_3 - f_1)$ and calculate $e\sin f_1$ as follows

$$q_3 = e\cos f_1 \cos(f_3 - f_1) - e\sin f_1 \sin(f_3 - f_1)$$

$$e\sin f_1 = \frac{q_1 \cos(f_3 - f_1) - q_3}{\sin(f_3 - f_1)}. \tag{13}$$

Equations (12) and (13) can then be solved simultaneously to yield e and f_1.

Having found the semi-lattice rectum and the eccentricity, we readily calculate the semi-major axis, a from the elliptical relation:

$$p = a(1 - e^2).$$

The mean motion, $\bar{n}$ follows directly from,

$$a^3\bar{n}^2 = k^2\mu$$

where $k\mu^{1/2}$ is just G, the constant of gravitation times the sum of the mass of the Earth and the satellite.

Next we can find the argument of latitude $u_1 = \omega + f_1$ from the equations

$$\sin u_1 = z_1/r_1 \sin i$$

$$\cos u_1 = (x_1 \cos\Omega + y_1 \sin\Omega)/r_1.$$

Then the argument of perigee is readily obtained from $\omega = (u_1 - f_1)$.

One element, the time of perigee passage remains to be found. If the mean anomaly, the angle swept out by a radius vector with mean angular velocity $\bar{n}$ in the time interval $(t - \tau)$ were known, we could calculate τ from the equation,

$$M_1 = \bar{n}(t_1 - \tau).$$

In order to calculate the mean anomaly, the eccentric anomaly must first be found. The true anomaly, already calculated for time t_1 and the eccentric anomaly, E_1 are related by,

$$\tan(E_1/2) = \tan(f_1/2)\left\{(1 - e)/(1 + e)\right\}^{1/2}.$$

In addition, from Figure 4 we see that,

$$r_1\cos f_1 = a\cos E_1 - ae.$$

The second equation can be used to determine the quadrant of E_1. Finally, Kepler's equation relates the mean and the eccentric anomalies,

$$M_1 = E_1 - e\sin E_1.$$

IV. DISCUSSION OF RESULTS & CONCLUSIONS

Two basic positions for the hunter satellite were assumed, one times (42, 164 km) and 1/4 times (10, 541 km) synchronous satellite altitude. The following orbital configurations were investigated:

	Hunter Satellite	Hunted Satellite
(a)	1 sync	1/4 sync
(b)	1/4 sync	1 sync
(c)	1 sync	3 sync

The two satellites were placed in various relative positions to each other in their respective orbits. PEP was then employed to compute the earth centered inertial coordinates of each satellite and the right ascension (R. A.) and declination (Decl.) look angles from the hunter to the hunted satellite all as a function of time. The quantities needed as input to the Gauss program are as follows:

Ephemeris Time	R.A.	Decl.	X	Y	Z
t_1	α_1	δ_1	X_1	Y_1	Z_1
t_2	α_2	δ_2	X_2	Y_2	Z_2
t_3	α_3	δ_3	X_3	Y_3	Z_3

The X, Y, Z are the hunter centered inertial coordinates of the Earth, just the negatives of the values computed by PEP,

which are assumed known for all times. In general, interpolation of the tabular values was necessary.

In the classic problem of Ceres, the planet moved a total of about 3° in 1 1/3 months of observation. In configuration (c) the time between observations was varied from 1 hr. (3°) to 6 hr. (18°). In general, the method failed in cases where the total arc length between r_1 and r_3 was greater than about 10°. Furthermore, in many configuration (a) situations there resulted two nearly equal positive roots of the equation in r_2, of which one leads to approximately correct orbital elements, but the other one does not. It was difficult to provide unambiguous instructions which told the computer which one of the two roots was the "correct" one to use. Many configuration (a) situations either failed outright or were very slow to converge. At first the poor success rate in configuration (a) was thought to be caused by lack of coordinate precision resulting in the use of linear interpolation. However, when a more accurate Hermite interpolation was used, the results were no better. The difficulty appears to be inherent in the physics of the situation, i.e., the lower the satellite orbit the more it deviates from the Keplerian approximation probably because of the non-spherical Earth perturbations.

Variation in the number of significant figures of the input data has shown that 5 place accuracy in ephemeris time and hunter inertial coordinates is sufficient for most preliminary orbit determinations. Angular errors of the order of $\pm 0.01°$ (36 arc seconds)* seem to be acceptable, but hunter satellite wobble might destroy this accuracy unless some way is found to make the necessary corrections. One way would be comparison with the local star field. Errors of the order of 1° in either angle result in complete failure of the method.

The following table summarizes the results in each of the orbital configurations. About 6 sets of observations apply in each of the cases averaged.

Given Orbital Element for Hunted Satellite

	a	e	i	Ω	ω	M
(a)	10,541	10^{-3}	10°	0°	90°	135°
(b)	42,164	0	0°	÷	÷	0°
(c)	126,500	10^{-3}	10°	0°	90°	0°

Average Errors in Calculations

	Δa	Δe	Δi	$\Delta\Omega$	$\Delta\omega$	ΔM
(a)	3220	0.13	0.33°	181°	95°	127°
(b)	3300	0.10	0.39°	÷	÷	-20°
(c)	6870	0.086	0.044°	0.91°	19.1°	-0.42°

* This accuracy could come from smoothing several observations.

÷ Indeterminate

As suggested above, the errors in the computed elements decrease with altitude, configuration (a) to (c), as the assumption of Keplerian motion becomes more accurate (at least out to 3 times synchronous altitude).

There is a so called neutral point on a line connecting the Earth-Moon centers where the two gravitational forces are equal and opposite. This distance is approximately 345,000 km from the Earth; as one approaches this distance, the lunar perturbations become dominant and the satellite motion again tends to become non-Keplerian.

In configuration (a) the errors in the last three elements are very large. The error in i is not bad but is still large enough to indicate difficulty. One would have great difficulty in locating such a satellite at some later time because of the large mean anomaly error. We find configuration (b) to yield more accurate results even though some of the numbers don't look much better than in (a). Actually this is a good test because e and i are both zero; therefore Ω and ω are not defined. The computed e and i, however, are not zero (hence the computed Ω and ω will not be zero). But with the computed inclination less than 1/2° from the celestial equator one should have little difficulty in locating the object again with a 6° field-of-view instrument (a reasonably conservative value) at a later time

even with a mean anomaly error of 20°. In configuration (c) all the errors are very much smaller than in either of the other configurations and very good preliminary orbit determination is realized in this case.

One example which has not been successfully evaluated is the highly eccentric orbit. It is not likely that one will encounter surreptitious satellites in such orbits because of the high risk of detection at perigee. Nevertheless, it is a possibility. We attempted to test the method on such an orbit but were unable to compute the necessary ephemerides because of the excessive computer time needed for the integrations. We did not investigate other possibly interesting orbit configurations because of the limited time alloted to this study.

We conclude that "reasonably accurate" determination of the orbit of one satellite from another by Gauss' method is feasible and practicable in most cases of interest with 5 place accuracy in the inertial coordinates and ephermis time and $\pm 0.01°$ accuracy in the look angles. Gauss' method is progressively more successful as the hunted satellite altitude is increased from 1/4 to 3 times synchronous. The problem, however, is a very complex one, and it is difficult to extrapolate these results to <u>all</u> cases of interest.

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APPENDIX

Gauss' Constant

Kepler's third law for a planet of mass m revolving about the sun of mass M may be stated as follows:

$$k^2(m + M)T^2 = 4\pi^2 a^3$$

where T is the period and a the semi-major axis. The value of k depends upon the units of time, distance and mass. Following Gauss, the units are taken to be the solar mass, the mean solar day and the Earth's mean distance from the Sun (the astronomical unit) in heliocentric problems. Kepler's third law becomes

$$K^2(1 + m)T^2 = 4\pi^2 a^3$$

The quantity k is called the Gaussian constant of gravitation and in the units defined above we have k = 0.01720209895.

For satellite motion about the Earth we can choose any convenient units: the ephemeris minute, the mass and radius of the Earth or the ephemeris day, the mass of the Earth and km, etc. To determine the value of k in this latter case we use Kepler's third law and take m to be the mass of the Moon in units of Earth masses, $\frac{1}{(81.31)}$, T to be 27.321661 days and a to be 377,028 km, the semi-major axis of the lunar orbit. Then the quantity k has a value 5.2915 E+7. If we use the astronomical unit instead of km, then k has a value 2.89125 E-5.